Theorem :- For any three sets A, B and C, Show that

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

Proof :- Let $a, b \in A \times (B \cup C)$

$\Rightarrow a \in A, b \in (B \cup C)$

$\Rightarrow a \in A, (b \in B \text{ or } b \in C)$

$\Rightarrow (a \in A, b \in B) \text{ or } (a \in A, b \in C)$

$\Rightarrow (a, b) \in (A \times B) \text{ or } (a, b) \in (A \times C)$

$\Rightarrow (a, b) \in (A \times B) \cup (A \times C)$

$\therefore A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ —— (i)

~~Now Let a, b~~

Again Let $a, b \in (A \times B) \cup (A \times C)$

$\Rightarrow (a, b) \in (A \times B) \text{ or } (a, b) \in (A \times C)$

$\Rightarrow (a \in A; b \in B) \text{ or } (a \in A, b \in C)$

$\Rightarrow a \in A, (b \in B \text{ or } b \in C)$

$\Rightarrow a \in A, b \in (B \cup C)$

$\Rightarrow a, b \in A \times (B \cup C)$

$\therefore (A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$ —— 2

From equation 1 and 2, We get

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

—— Proved.

Theorem :- If A, B and C be three sets ~~and~~ Such that $A \neq \phi$, $B \neq \phi$ and $(A\times B) \cup (B\times A) = C\times C$, then show that $A = B = C$.

Proof:- Given that $A \neq \phi$, $B \neq \phi$, then

$A\times B \neq \phi$ and $B\times A \neq \phi$

$\therefore C\times C = (A\times B) \cup (B\times A) = \phi$

Let $a \in C$

$\Rightarrow (a,a) \in C\times C$

$\Rightarrow (a,a) \in (A\times B) \cup (B\times A)$ given

$\Rightarrow (a,a) \in (A\times B)$ or $(a,a) \in B\times A$

$\Rightarrow a \in A,\ a \in B$

Now $a \in C \Rightarrow a \in A$ $\quad \therefore C \subseteq A$

In the same way $a \in C \Rightarrow a \in B$ $\therefore C \subseteq B$

$\therefore C \subseteq A$ and $C \subseteq B$ ———— (i)

Again $A = \phi$, $B = \phi$ then If $a \in A$, $b \in B$

$\Rightarrow (a,b) \in (A\times B)$

$\Rightarrow (a,b) \in (A\times B) \cup (B\times A)$ $[\because a \in X \Rightarrow a \in X \cup Y]$

$\Rightarrow (a,b) \in C\times C$ (given)

$\Rightarrow a \in C,\ b \in C$

$\therefore A \subseteq C$ and $B \subseteq C$ ———— (ii)

By eqn (1) and (ii)

$A = C,\ B = C \Rightarrow A = B = C$

Proved.